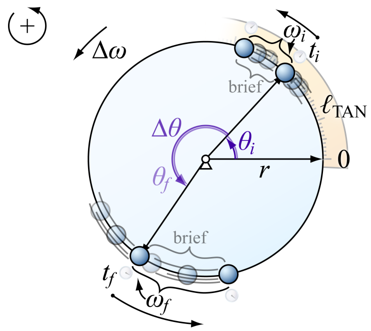
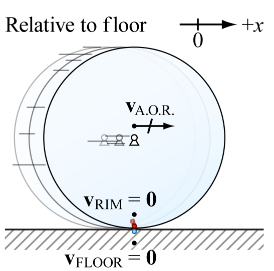
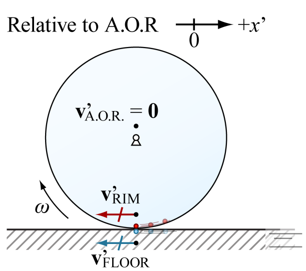
Rotational kinematics and dynamics (calculus-based physics)

Kinematics



Angular	Tangential	If non-slip
$\theta := \dfrac{\ell_{TAN}}{r}$	$\Delta\ell_{TAN} = r\Delta\theta$	$\Delta\ell_{A.O.R.} = \Delta\ell_{TAN}$
$\omega_{\circlearrowleft} := \dfrac{d\theta}{dt}$	$v_{TAN} = r\omega$	$v_{A.O.R.} = v_{TAN}$
$\alpha := \dfrac{d\omega}{dt}$	$a_{TAN} = r\alpha$	$a_{A.O.R.} = a_{TAN}$





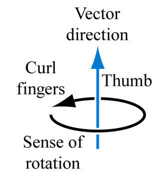
Relationships for UαM

$\theta_i + \omega_{AVG}\,\Delta t = \theta_f$	α
$\omega_i + \alpha_{AVG}\,\Delta t = \omega_f$	θ
$\omega_{AVG} = \dfrac{\omega_i + \omega_f}{2}$	t, θ, α
$\theta_i + \omega_i\Delta t + \dfrac{1}{2}\alpha\Delta t^2 = \theta_f$	
$\omega_i^2 + 2\alpha\Delta\theta = \omega_f^2$	t

Dynamics

Rotational vectors

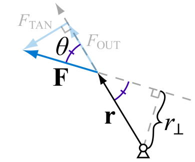
RHR



Torque

$$\vec{\boldsymbol{\tau}}_F := \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

$$\tau_F = \pm r_\perp F = \pm(r\sin\theta)F$$



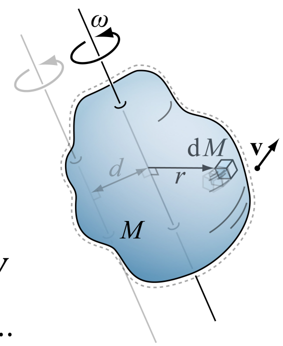
Rotational inertia

$$I_{\substack{\text{RIGID SET}\\ \text{OF}\\ \text{PARTICLES}}} := \sum_i \Delta M_i r_i^2$$

$$I_{\substack{\text{RIGID}\\ \text{CONTINUOUS}\\ \text{MASS DISTR.}}} := \int r^2\,dM = \int r^2\rho\,dV$$

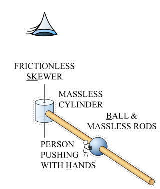
$$I_{\substack{\text{RIGID}\\ \text{PARTS}}} = I_1 + I_2 + I_3 + \cdots$$

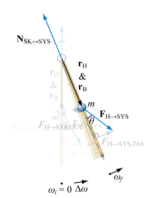
$$I_{\parallel} = I_{C.O.M.} + Md^2$$



Newton's 2nd law for rotation

$$\vec{\boldsymbol{\alpha}} = \frac{\sum \vec{\boldsymbol{\tau}}}{I}$$





Summing torques

1. Draw spatially-extended **free-body diagram** with the **tail** of each force vector anchored at its **point of application**.
2. Draw $+x$ and $+y$ directions, **axis of rotation**, and positive **sense of rotation**.
3. Fill in $\sum\tau = I\alpha$, determining the **sign** of each τ by considering whether each force, in isolation, would spin up the object in the ccw or cw direction.

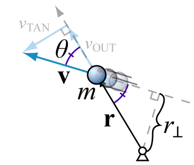
Conservation laws

Angular momentum

$$\vec{\mathbf{L}}_{PARTICLES} := \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i$$

$$L_{PARTICLE} = \pm m v r_\perp = \pm(r\sin\theta)p$$

$$\vec{\mathbf{L}}_{RIGID} = I_{\substack{\text{ABOUT}\\ \text{FIXED}\\ \text{SKEWER}}}\,\vec{\boldsymbol{\omega}}$$



$$\vec{\mathbf{L}}_{RIGID} = \vec{\mathbf{L}}_{\substack{\text{C.O.M.}\\ \text{ORBITS}\\ \text{ORIGIN}}} + \vec{\mathbf{L}}_{\substack{\text{SPIN}\\ \text{ABOUT}\\ \text{C.O.M.}}}$$

$$\frac{d\vec{\mathbf{L}}}{dt} = \Sigma\,\vec{\boldsymbol{\tau}}$$

$$\Sigma\vec{\mathbf{L}}_i + \int_{t=t_i}^{t=t_f}\left(\sum_{\text{EXT}\to\text{SYS}} \vec{\boldsymbol{\tau}}\right)dt = \Sigma\vec{\mathbf{L}}_f$$

Summing angular momenta

1. Illustrate **before** and **after** situations.
2. Draw **axis of rotation**.
3. Draw positive **sense of rotation**.
4. Determine **sign** of each object's L by determining whether rotation is ccw or cw.

Energy

$$K_{PARTICLES} := \sum_i \frac{1}{2}\Delta M_i v_i^2$$

$$K_{\substack{\text{CONTINUOUS}\\ \text{MASS DISTR.}}} := \int \frac{1}{2}v^2\,dM$$

$$K_{RIGID} = \frac{1}{2} I_{\substack{\text{ABOUT}\\ \text{FIXED}\\ \text{AXIS}}}\,\omega^2 \qquad \Delta W_\tau = \int_{\theta=\theta_i}^{\theta=\theta_f} \tau\,d\theta$$

$$K_{RIGID} = \frac{1}{2}Mv_{C.O.M.}^2 + \frac{1}{2}I_{\substack{\text{ABOUT}\\ \text{C.O.M.}}}\,\omega^2$$